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Computational Integrated-Block-Based Simulation Developed to Approximate Solutions of Multi-Degree Regular Differential Equations

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ABSTRACT

This paper introduces a novel unified numerical approach to obtain explicit approximations of initial value formulations (IVFs) encompassing ODEs ranging from first- to third-order. The proposed technique leverages Chebyshev polynomials as basic functions and is developed using continuous schemes formulated through both collocation and interpolation strategies. It operates on a block-by-block basis, providing an efficient framework for numerically solving ODEs of multiple orders.

The convergence properties of this method are thoroughly examined through the lens of zero-stability and consistency. In-depth discussions unfold, shedding light on the efficacy of this approach in addressing first, second, and third-order ODEs. Through comparative analyses against existing methods, it is distinctly evident that the proposed model surpasses its counterparts in terms of accuracy, marking a significant advancement in the numerical treatment of IVPs. This model not only introduces a unified approach for diverse ODE orders but also stands as a testament to its superior performance, establishing itself as a noteworthy contribution to the realm of numerical integration methodologies.

Keywords: Unified Numerical Approach; Initial Value Formulations (IVFs); Ordinary Differential Equations (ODEs); Chebyshev Polynomials; Collocation; Interpolation; Block Method; Zero-Stability; Consistency; Convergence Analysis; Numerical Integration; Accuracy.

1. Introduction

Numerical methods have become a widely adopted and popular approach for approximating the solutions of differential equations. It is well established that many real-world problems, when formulated as initial value ordinary differential equations, lack analytical solutions [1, 2]. Consequently, the ongoing search for more efficient computational and numerical approximation methods has remained essential over the years.

In this work, we consider the initial condition problems related to ODEs represented by the expressions:

$$(x) = f(x, \mu(x)) \quad \dots(1)$$

with initial conditions,

$$\mu(x_0) = \mu_0,$$

$$\mu''(x) = f(x, \mu(x), \mu'(x)) \quad \dots(2)$$

with initial conditions,

$$\mu(x_0) = \mu_0, \mu'(x_0) = \mu_1,$$

$$\mu'''(x) = f(x, \mu(x), \mu'(x), \mu''(x)) \quad \dots(3)$$

with initial conditions,

$$\mu(x_0) = \mu_0, \mu'^{(x_0)} = \mu_1, \mu''(x_0) = \mu_2,$$

Several authors have suggested various numerical methods for the solutions of equations (1), (2) and (3) [3, 4, 5], developed a non-standard finite method for the solutions of (1) and [5] proposed the predictor-corrector method for its solution. [6, 7, 8, 9] and many others have developed self-starting implicit continuous linear multistep methods by implementing collocation strategy alongside interpolation of various basis approximation solutions such as power series for the direct solution of (2). Also, authors in [10, 11, 12, 13] have equally proposed direct methods of approximate solutions to equations of the form (3).

It is worthy of note that the development of a single scheme for addressing equations expressed in the structure (1), (2) & (3) is a recent phenomenon that has attracted attention of many researchers in the current research era.

The uniqueness of the method proposed in this work is in its capability to solve first and higher order of IVPs ODEs with better efficiency and higher rate of convergence using a single scheme. An analysis of its fundamental stability properties indicating that the technique maintains zero-stability, ensures consistency, and achieves convergence. To evaluate its effectiveness, we applied it to first-, second-, and third-order IVP ODE problems. The results demonstrated that this new approach is not only computationally reliable but also more cost-effective compared to existing numerical methods.

1.1. Study Objectives

The following are the objectives of this study:

1. Our goal is to create a cutting-edge simulation framework that uses an integrated block-based approach to find solutions for multi-degree regular differential equations, all while boosting accuracy and efficiency.

2. We aim to implement numerical schemes that utilize block-based methods, allowing us to tackle differential equations of different degrees in a cohesive way by breaking the problem into manageable stages.

3. We will thoroughly analyze the stability, consistency, and convergence of our proposed simulation method, applying well-established criteria from numerical analysis.

4. We plan to compare how our integrated block-based method stacks up against traditional single-step and multi-step numerical techniques, focusing on accuracy, computational costs, and minimizing errors.

5. Our objective is to assess how adaptable our method is for solving first-, second-, and higher-order regular differential equations, all within a single computational framework.

6. Finally, we will showcase the real-world applicability of our simulation through benchmark problems and case studies drawn from various fields in science and engineering.

2. Material and Method

Here, the approximate polynomial is chosen as

$$y(x)=\sum_{j=0}^{k+7}a_jT_j(x) \qquad \ldots(4)$$

Here, $T_j(x)$ denotes the Chebyshev polynomial and $a_j's$ are unknown coefficients to be determined. Equation (4) undergoes interpolation evaluation at the point $x=x_n$ while the derivatives of the first and second degree are obtained by computing (determined at collocation points) at the fractional nodes $x = x_{n+v}, v = \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$. The third derivative is obtained and collocated at $x = x_{n+w}, w = 1$. Thus, we obtain the generalized expression:

$$\begin{aligned} \sum_{j=0}^{k+7} a_j T_j &= y_n \\ \sum_{j=1}^{k+7} a_j T_j' &= f_{n+v} \\ \sum_{j=2}^{k+7} a_j T_j'' &= g_{n+v} \\ \sum_{j=3}^{k+7} a_j T_j''' &= K_{n+w} \end{aligned} \qquad \ldots(5)$$

where; f_{n+v} is the first derivative of (4), g_{n+v} is its second derivative and K_{n+w}, is third derivative and where k denotes the step index, with k=1. a_j's

This unknowns a_j's' are determined through the application of Gaussian elimination and then inserted into Eq. (4), resulting in a continuous implicit formulation.

$$\alpha_0(t) y_n = h\left(\sum_{j=0}^{1} \beta_{\frac{3}{7}}(t) f_{n+\frac{3}{7}} + \beta_{\frac{4}{7}}(t) f_{n+\frac{4}{7}} + \beta_{\frac{5}{7}}(t) f_{n+\frac{5}{7}}\right) + h^2\left(\sum_{j=0}^{1} \lambda_{\frac{3}{7}}(t) g_{n+\frac{3}{7}} + \lambda_{\frac{4}{7}}(t) g_{n+\frac{4}{7}} + \lambda_{\frac{5}{7}}(t) g_{n+\frac{5}{7}}\right) + h^3\left(\delta_1(t) K_{n+1}\right) \qquad \ldots(6)$$

Where, $t = \frac{2x - 2x_n - h}{h}$, $\alpha_0(t) = 1$ and

$$\begin{pmatrix} \beta_0 \\ \beta_{\frac{3}{6}} \\ \beta_{\frac{4}{6}} \\ \beta_{\frac{5}{6}} \\ \lambda_0 \\ \lambda_{\frac{3}{6}} \\ \lambda_{\frac{4}{6}} \\ \lambda_{\frac{5}{6}} \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & -8 & 18 & -32 & 50 & -72 & 98 & -128 & 162 \\ 0 & 2 & \frac{-8}{7} & \frac{-270}{49} & \frac{1504}{343} & \frac{18290}{2401} & \frac{-154440}{16807} & \frac{-127762}{16807} & \frac{12134272}{823543} & \frac{29040930}{5764801} \\ 0 & 2 & \frac{8}{7} & \frac{-270}{49} & \frac{-1504}{343} & \frac{18290}{2401} & \frac{-154440}{16807} & \frac{-127762}{16807} & \frac{12134272}{823543} & \frac{29040930}{5764801} \\ 0 & 2 & \frac{24}{7} & \frac{-78}{49} & \frac{-2976}{343} & \frac{-15950}{2401} & \frac{103896}{16807} & \frac{260206}{16807} & \frac{5702016}{823543} & \frac{-76262238}{5764801} \\ 0 & 0 & 16 & -96 & 320 & -800 & 1680 & -3136 & 5376 & -8640 \\ 0 & 0 & 16 & \frac{-96}{7} & \frac{-2752}{49} & \frac{22240}{343} & \frac{236688}{2401} & \frac{-399808}{2401} & \frac{-1879296}{16807} & \frac{260518464}{823543} \\ 0 & 0 & 16 & \frac{96}{7} & \frac{-2752}{49} & \frac{-22240}{343} & \frac{236688}{2401} & \frac{399808}{2401} & \frac{-1879296}{16807} & \frac{-260518464}{823543} \\ 0 & 0 & 16 & \frac{288}{7} & \frac{320}{49} & \frac{-36000}{343} & \frac{359280}{2401} & \frac{62784}{2401} & \frac{4973312}{16807} & \frac{232960320}{823543} \\ 0 & 0 & 0 & 192 & 1536 & 6720 & 21504 & 56448 & 129024 & 266112 \end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \end{pmatrix}$$

Evaluating Eq. (6) at $x = x_{n+1}(t = 1)$, $x = x_{n+\frac{5}{7}}(t = \frac{3}{7})$, $x = x_{n+\frac{4}{7}}(t = \frac{1}{7})$ and $x = x_{n+\frac{3}{7}}(t = \frac{-1}{7})$ yields

$$\begin{pmatrix} y_{n+\frac{3}{7}} \\ y_{n+\frac{4}{7}} \\ y_{n+\frac{5}{7}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y_n + hD \begin{pmatrix} f_n \\ f_{n+\frac{3}{7}} \\ f_{n+\frac{4}{7}} \\ f_{n+\frac{5}{7}} \\ f_{n+1} \end{pmatrix} + h^2 E \begin{pmatrix} g_n \\ g_{n+\frac{3}{7}} \\ g_{n+\frac{4}{7}} \\ g_{n+\frac{5}{7}} \\ g_{n+1} \end{pmatrix} + h^3 F \begin{pmatrix} T_n \\ T_{n+\frac{3}{7}} \\ T_{n+\frac{4}{7}} \\ T_{n+\frac{5}{7}} \\ T_{n+1} \end{pmatrix} \quad \ldots(7)$$

Where

$$D = \begin{pmatrix} \frac{18177619113}{141786400000} & \frac{17256121762}{134586309375} & \frac{1413747103}{11025310464} & \frac{103181415259}{703144800000} \\ \frac{-620083203}{141786400} & \frac{-4640727776}{1076690475} & \frac{-5927626625}{1378163808} & \frac{2991267671}{703144800} \\ \frac{2921981283}{1134291200} & \frac{105711646}{39877425} & \frac{1113069875}{408344832} & \frac{971248747}{208339200} \\ \frac{37177940439}{17723300000} & \frac{10480166944}{4984678125} & \frac{110322227}{51043104} & \frac{-26246614849}{26246614849} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} \frac{463348677}{99250480000} & \frac{293266132}{62806944375} & \frac{60071845}{12862862208} & \frac{142979891}{23438160000} \\ \frac{-9575151}{28357280} & \frac{-24145024}{71779365} & \frac{154341875}{459387936} & \frac{4515889}{46876320} \\ \frac{-2312396019}{3970019200} & \frac{-163415956}{279141975} & \frac{-831335875}{1429206912} & \frac{118384147}{104169600} \\ \frac{-325377}{1904000} & \frac{-206272}{1204875} & \frac{-213125}{1233792} & \frac{-1728}{14875} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{3807}{248126200} & \frac{4288}{279141975} & \frac{1375}{89325432} & \frac{1409}{6510600} \end{pmatrix}.$$

Hence, we show the investigation of the basic properties in the next section.

3. Assessment of the Methodology

This section explores the core characteristics of the proposed numerical technique. The investigation includes its order of accuracy, error constant, zero-stability, consistency, and computational performance. Each property is systematically analyzed to validate the robustness of the scheme.

3.1. Degree of Accuracy and Associated Error Coefficient

This formulation is classified as a multi-stage numerical approach and refers to in the form:

$$\sum_{j=0}^{k}\alpha_j(t)y_{n+j}=h\left(\sum_{j=0}^{k}\beta_j(t)f_{n+j}\right)+h^2\left(\sum_{j=0}^{k}\lambda_j(t)g_{n+j}\right)+h^3\left(\sum_{j=0}^{k}\delta_j(t)T_{n+j}\right) \quad ...(8)$$

As outlined in references [9] and [11], the expression for the associated truncation inaccuracy (LTE) of Eq. (8) is expressed using a difference operator as follows:

$$L[y(x):h]=\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh)-h^2\beta_j f(x_n+jh)-h^3\gamma_j g(x_n+jh)\right] \quad ...(9)$$

Where f (x,y) represents a function possessing continuous derivatives throughout the closed interval [a, b]. By Expanding Equation (9) around x using the Taylor series, we obtain the expression below

$$L[y(x);h]=C_0y(x)+C_1hy'(x)+C_2h^2y''(x)+\ldots+C_{p+2}h^{p+2}y^{p+2}(x)$$

with the condition that the c_0 , C_1 , C_2 … c_p … c_{p+2} are obtained as $c_0=\sum_{j=0}^{k}\alpha_j$, $c_1=\sum_{j=1}^{k}j\alpha_j$,

$$C_2=\frac{1}{2!}\sum_{j=1}^{k}j^2\alpha_j, \qquad C_q=\frac{1}{q!}\left[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)\sum_{j=1}^{k}\beta_j j^{q-2}-q(q-1)(q-2)\sum_{j=1}^{k}\gamma_j j^{q-3}\right].$$

In the spirit of [14], Eq. (9) is of order p if $C_0=C_1=C_2=\ldots C_p=C_{p+1}=0$ and $C_{p+r}\neq 0$. The $C_{p+r}\neq 0$ is called the error constant and $C_{p+r}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

Thus, block (7) is of order $p=8$ and error constant

$$C_{p+2}=\left[\frac{162948899}{3204091891585088000},\frac{232193044}{4562076150323299125},\frac{9520391}{18686263911724233216},\frac{242225}{13897679476336000}\right]^T.$$

3.2. Zero Stability of the Method

A linear multistep technique is considered zero-stable provided that every root of its primary characteristic polynomial R(ρ) lies within or on the unit circle in the complex plane. Furthermore, any root lying exactly on the unit circle must have a multiplicity that does not exceed the differential equation's order [5].

To assess whether the developed scheme is zero-stable, we rewrite equation (8) using a vector-based notation. Let us denote the vector:

$$e=(e_1\ldots e_r)^T,\quad d=(d_1\ldots d_r)^T,\quad y_m=(y_{n+1}\ldots y_{n+r})^T,\ F(y_m)=(f_{n+1}\ldots f_{n+r})^T,\ G(y_m)=(g_{n+1}\ldots g_{n+r})^T,$$

and matrices $A=(a_{ij})$, $B=(b_{ij})$.

Thus, Eq. (7) forms the block formula

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$$A^0 y_m = hBF(y_m) + A^1 y_n + hbf_n + h^2 DG(y_m) + h^2 dg_n + h^3 VW(y_m) + h^3 uT_n \quad \ldots(10)$$

where h represents a constant step size within the block.

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \frac{18177619113}{141786400000} \\ \frac{-620083203}{141786400} \\ \frac{2921981283}{1134291200} \\ \frac{37177940439}{17723300000} \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} \frac{463348677}{99250480000} \\ \frac{-9575151}{28357280} \\ \frac{-2312396019}{3970019200} \\ \frac{-325377}{1904000} \\ 0 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{3807}{248126200} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{17256121762}{134586309375} & \frac{1413747103}{11025310464} & \frac{103181415259}{703144800000} \\ \frac{-4640727776}{1076690475} & \frac{-5927626625}{1378163808} & \frac{2991267671}{703144800} \\ \frac{105711646}{39877425} & \frac{1113069875}{408344832} & \frac{971248747}{208339200} \\ \frac{10480166944}{4984678125} & \frac{110322227}{51043104} & \frac{-26246614849}{26246614849} \\ 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{293266132}{62806944375} & \frac{60071845}{12862862208} & \frac{142979891}{23438160000} \\ \frac{-24145024}{71779365} & \frac{154341875}{459387936} & \frac{4515889}{46876320} \\ \frac{-163415956}{279141975} & \frac{-831335875}{1429206912} & \frac{118384147}{104169600} \\ \frac{-206272}{1204875} & \frac{-213125}{1233792} & \frac{-1728}{14875} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{4288}{279141975} & \frac{1375}{89325432} & \frac{1409}{6510600} \end{pmatrix}$$

Considering the initial defining equation of the hybrid block formulation approach, as defined by the characteristic's equation

$$\rho(R) = \det\,(RA^0 - A^1) \quad \ldots(11)$$

Where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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Substituting A^0 and A^1 as specified in equation (11), the values of R, produces the resulting values 0,0,0 and1.

As stated in [11], the block method in equation (7) exhibits zero-stability, as R(ρ)=0 meets the condition |Rj|≤1 for j=1, and the multiplicity of roots where the magnitude of |Rj| equals 1is at most two.

3.3. Method Consistency and Convergence Analysis

A linear-based multi-step scheme, as defined by equation (8), is deemed consistent if it possesses a positive order $p \geq 1$. Eq. (7) has an order of 8.

Based on the convergence theorem proved by [4], a multi-step numerical method of linear form achieves convergence only when it satisfies both consistency and zero-stability conditions. Since the proposed scheme satisfies both these properties, thus, the method satisfies the conditions for convergence.

3.4. Numerical Experiments

This section presents real-world implementations of the newly developed technique. A range of ordinary differential equations spanning IVPs involving Equations of the first, second, and third order derivatives are addressed to evaluate the method's precision and effectiveness.

Problem 1: Given the conditions

$y(0) = 0.5, step\ size\ h = 0.1, and\ the\ differential\ equation\ y' = 0.5(1 - y)$

Analytical Solution: $y(x) = 1 - 0.5e^{-0.5x}$

Table 1. An evaluation of the error generated by the newly introduced block scheme in contrast with traditional methods for solving Problem 1 is performed

x- values	Error in new method	Error in [15]	Error in [3]	Error in [16]
0.1	8.30000000E-24	1.000000E-10	1.218026E-13	5.574430E-12
0.2	1.56000000E-23	1.000000E-10	1.399991E-13	3.946177E-12
0.3	2.23000000E-23	1.000000E-10	1.184941E-12	8.183232E-12
0.4	2.82000000E-23	2.000000E-10	1.538991E-12	3.436118E-15
0.5	3.35000000E-23	3.000000E-10	1.110001E-12	1.929743E-10
0.6	3.83000000E-23	3.000000E-10	5.270229E-12	1.879040E-10
0.7	4.25000000E-23	2.000000E-10	2.108980E-12	1.776835E-10
0.8	4.63000000E-23	3.000000E-10	1.297895E-11	1.724676E-10
0.9	4.95000000E-23	3.000000E-10	3.082290E-11	1.847545E-10
1.0	5.23000000E-23	2.000000E-10	4.121925E-11	3.005770E-10

Problem 2: $y(0) = 1, y'(0) = 0, y''(0) = 0, and\ the\ third - order\ differetial\ equation\ y''' = 3\sin(x)\ with\ a\ step\ size\ h = 1.0$

Analytical Solution: $y(x) = 1 - \frac{3}{2}\cos(x)$

Table 2. A comparative analysis of the errors generated by the proposed block method and those from existing techniques in solving Problem 2 is presented

x- values	Error in the new method	Error in [15]	Error in [8]	Error in [13]
0.1	5.221310E-020	2.000000E-010	6.370460E-13	1.65922E-10
0.2	1.038925E-019	4.000000E-010	4.052980E-12	4.76275E-10
0.3	1.545210E-019	2.000000E-010	1.009326E-11	6.23182E-10
0.4	2.035925E-019	2.000000E-010	1.890366E-11	19.9134E-10
0.5	2.506192E-019	9.000000E-010	3.033807E-11	3.28882E-10
0.6	2.951289E-019	1.100000E-009	4.455258E-11	1.27096E-09
0.7	3.366768E-019	1.500000E-009	5.987466E-11	4.84653E-09
0.8	3.748489E-019	1.300000E-009	7.711903E-11	1.09585E-08
0.9	4.092638E-019	1.500000E-009	9.618412E-11	2.01880E-08
1.0	4.395763E-019	2.000000E-009	1.171654E-10	3.53956E-08

Problem 3: Consider the initial conditions

$y'' = y', \ \ y'(0) = 0 \ and \ y(0) = -1 \ with \ step \ size \ h = 0.1$

Analytical Solution: $y(x) = 1 - e^x$

Table 3. An assessment of the discrepancy linked to the newly developed block technique in contrast with existing methods for solving Problem 3 is carried out

x- values	Error in the new method	Error in [2]	Error in [8]	Error in [17]
0.1	4.600000E-025	2.095826E-010	2.508826E-13	2.858824E-15
0.2	-8.600000E-026	2.092718E-009	6.493175E-11	1.439682E-12
0.3	-5.000000E-026	7.842546E-009	1.683146E-09	5.591383E-11
0.4	-2.500000E-025	2.009500E-008	1.700635E-08	4.796602E-09
0.5	-3.700000E-025	4.199771E-008	1.025454E-07	1.003781E-08
0.6	-1.300000E-025	7.728842E-008	2.558711E-06	1.590163E-08
0.7	-6.000000E-026	1.303844E-007	5.273300E-06	2.870014E-08
0.8	-2.500000E-025	2.064839E-007	8.275935E-06	4.284730E-08
0.9	-3.800000E-025	3.116817E-007	1.161667E-05	5.857869E-08
1.0	-2.000000E-025	4.531001E-007	1.542187E-05	8.449297E-08

Problem 4: Given the conditions

$y(0) = 3, y' = 1, y''(0) = 5, and \ the \ third - order \ differential \ equation \ y''' = e^x, with \ step \ size \ of \ h = 1.0$

Analytical Solution: $y(x) = 2 + 2x^2 + e^x$

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Table 4. A review of the error behavior exhibited by the introduced block-based approach, compared with known methods for solving Problem 4, is performed

x- values	Error in the new method	Error in [2]	Error in [17]	Error in [13]
0.1	-1.83670E-020	8.881784E-015	3.369305E-12	9.24352E-10
0.2	-3.86660E+020	3.552714E-014	2.160050E-11	8.39830E-10
0.3	-6.11000E+020	8.304468E-014	5.333245E-11	4.23997E-10
0.4	-8.58930E-020	1.527667E-013	9.988632E-11	3.58729E-10
0.5	-1.13293E-019	2.460254E-013	1.598988E-10	2.99872E-10
0.6	-1.43575E-019	3.668177E-013	2.511404E-10	3.90509E-10
0.7	-1.77043E-019	5.178080E-013	3.961489E-10	1.47048E-09
0.8	-2.14029E-019	7.025491E-013	5.926823E-10	2.49247E-09
0.9	-2.54906E-019	9.254819E-013	8.429168E-10	0.15695E-09
1.0	-3.00083E-019	1.187495E-012	1.144603E-09	3.54096E-09

4. Discussion of Results

The tables provided display the computed results derived from applying the newly constructed method. Clearly, the proposed hybrid approach yields lower error margins than existing techniques, even with the relatively large step number k utilized in this study.

5. Conclusion

This study presents the creation and execution of a novel combined block approach specifically designed for addressing ordinary differential equations (ODEs) of the of the first, second, and third degrees. The proposed approach achieves an eighth-order accuracy, reflecting both its high level of precision and its compliance with consistency conditions.

One of the main advantages of this approach is its versatility it successfully handles various orders of ODEs within a single unified framework. The conducted simulations confirm the improved effectiveness of the proposed method compared to existing techniques, particularly in minimizing numerical error. As demonstrated in Tables I–IV, the approach delivers greater precision and improved computational performance.

6. Future Suggestions

1. Extension to fractional-order equations: Let's expand the method to cover fractional-order and singularly perturbed differential equations, broadening its reach.

2. Parallel and GPU implementation: Boost computational speed by tapping into high-performance and GPU-based computing resources.

3. Adaptive block sizing: Implement adaptive step or block size control to enhance accuracy and efficiency, especially when dealing with varying problem conditions.

4. Expansion to partial differential equations (PDEs): Widen the scope to include multi-dimensional PDEs, making it easier to tackle complex real-world challenges.

5. Automated error control mechanisms: Create built-in error estimation and correction features to enhance numerical stability and reliability.

Declarations

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Competing Interests Statement

The authors affirm that they have no competing interests that may have affected the findings or interpretations outlined in this study.

Consent for publication

The authors declare that they consented to the publication of this study.

Authors' contributions

All the authors made an equal contribution in the Conception and design of the work, Data collection, Drafting the article, and Critical revision of the article. All the authors have read and approved the final copy of the manuscript.

Institutional Review Board Statement

Not applicable for this study.

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