

Derivation and Implementation of an Equally Spaced Numerical Scheme (ESNS) for Solving Nonlinear Problems Modeled Differential Equations

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DOI: <https://doi.org/10.46431/mejast.2026.9305>

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Article Received: 08 July 2026

Article Accepted: 15 September 2026

Article Published: 24 September 2026

ABSTRACT

This paper presents the development and comprehensive analysis of an Equally Spaced Numerical Scheme (ESNS) for solving nonlinear modeled oscillatory differential equations, which frequently arise in engineering and biological systems where closed-form analytical solutions are unavailable. The ESNS is constructed using a continuous power series approximation evaluated on uniformly distributed grid points, a strategy that significantly enhances computational simplicity while improving stability and convergence characteristics. The derivation employs interpolation and collocation techniques, yielding a linear multistep method with a continuous form that is subsequently discretized to produce an efficient step-by-step numerical solver. A rigorous theoretical investigation of the scheme is conducted, including determination of its order and error constant, demonstration of consistency and zero-stability, proof of convergence, and delineation of its absolute stability region, which confirms that the method is A-stable and therefore suitable for stiff oscillatory problems. The practical performance of the ESNS is evaluated through numerical simulations on three classes of oscillatory differential equations, including a susceptible-infected-recovered (SIR) epidemiological model that computes disease transmission dynamics within a population over time. The numerical results are systematically compared with established methods, and both tabular and graphical comparisons demonstrate that the ESNS yields approximate solutions of remarkably high accuracy, characterized by extremely small error magnitudes. The findings confirm that the ESNS is a reliable, computationally efficient and robust numerical tool for solving nonlinear oscillatory initial value problems, offering a valuable alternative for researchers and practitioners in applied mathematics and related disciplines.

Keywords: Equally Spaced Numerical Scheme (ESNS); Nonlinear Differential Equations; Oscillatory Differential Equations; Stability Analysis; Differential Equations; Power Series Approximation; A-Stable Method; Convergence Analysis; Block Hybrid Method; Computational Efficiency.

1.0. Introduction

The paper emphasized the importance of developing efficient numerical techniques for solving nonlinear modeled differential equations, particularly when analytical solutions are unavailable. It introduced the Equally Spaced Numerical Scheme (ESNS) as a structured approach that uses uniformly distributed grid points to enhance stability, accuracy, and convergence while reducing computational complexity. Given the limitations of existing numerical methods such as instability, high computational cost, and sensitivity to step-size variations, the ESNS is derived as a simple and reliable alternative for nonlinear initial value problems commonly encountered in engineering and applied sciences. Additionally, the Natural Decomposition Method (NDM), developed by Rawashdeh & Maitama (2015), developed analytical-numerical technique that combined decomposition strategies to effectively handle nonlinear differential equations.

The paper consider the numerical simulation of nonlinear problems modelled as oscillatory class of differential equation of the form

$$\gamma(\xi) = f(\xi, \gamma), \gamma(\vartheta_0) = \varphi_0 \quad (1)$$

The Natural Transform Method with the Adomian Decomposition Method produced fast-converging series solutions that solve nonlinear ordinary differential equations and recover all exact solutions for particular test cases. Mengesha & Denekeew (2020), conducted the study on how to determine the solutions of nonlinear second-order

differential equations which was accomplished by developing new methods that introduced substitution techniques and adaptations of linear constant-coefficient methods. Also, Taylor series approximations were derived either direct solutions or indirect solutions for specific nonlinear problems. Chen & Feng, (2025) which, created a new continuous hybrid block linear multistep method which solves nonlinear second-order initial value problems directly without requiring conversion to first-order systems. The paper demonstrated through theoretical analysis and numerical experiments that the method achieves consistency and zero-stability and converges to highly precise results. The research studies demonstrate how scholars achieved both analytical and numerical progress to effectively solve nonlinear ordinary differential equations (Mengesha & Deneke, 2020; Rawashdeh & Maitama, 2015).

The literature on numerical methods for ordinary differential equations (ODEs) has witnessed significant advancements in hybrid block methods, which combine multiple intra-step points and derivative information to enhance accuracy, stability, and computational efficiency. Rufai et al. (2026a) developed an overlapping one-step multiderivative hybrid block method achieving Order 12, demonstrating that overlapping grid structures substantially reduce local truncation error and expand stability regions. Extending this approach, Rufai et al. (2026b) developed adaptive step-size selection mechanisms that automatically adjust computational parameters based on error estimation, significantly improving efficiency for problems with rapidly changing solutions. Other notable contributions include Zarnan et al. (2022), who combined numerical inverse Laplace transform with Chebyshev polynomials to solve nonlinear differential equations, and Shateyi (2023), who demonstrated that block hybrid methods with higher collocation point counts achieve exceptional accuracy on chaotic systems such as the Lorenz and Glukhovsky-Dolzansky models. Chauhan and Khirsariya (2023) developed the Adomian Decomposition General Transform Method for nonlinear fractional equations, establishing rigorous convergence analysis in Banach spaces, while Sabo et al. (2024) developed a block hybrid numerical method with uniform order seven that demonstrated strong absolute stability for stiff computational challenges. Collectively, these studies underscore the growing recognition that hybrid block methods, particularly those incorporating adaptive strategies and multiderivative information, represent a promising direction for solving complex initial value problems.

The treatment of oscillatory differential equations has received considerable attention due to their prevalence in engineering, physics, and biological systems, where solutions exhibit periodic behavior that demands specialized numerical treatment. Sunday et al. (2025) formulated a self-starting three-step algorithm for direct computation of second-order oscillatory problems, deliberately avoiding reduction to first-order systems to preserve the structural integrity of the original equations. Their comprehensive theoretical investigation examined order, consistency, zero-stability, convergence, and stability region, demonstrating that the method achieves high accuracy with error magnitudes substantially smaller than conventional approaches. The broader theoretical foundations have been rigorously examined by Ding et al. (2026), who proved that linear and quadratic finite element methods possess A-stability for autonomous stiff ODEs, ensuring reliable performance even with relatively large step sizes. Additionally, Sharifi et al. (2026) introduced second derivative diagonally implicit two-step peer methods with Runge-Kutta stability, contributing to the expanding family of A-stable techniques suitable for stiff oscillatory problems.

The application of numerical methods to epidemiological modeling, particularly the susceptible-infected-recovered (SIR) framework, has gained significant momentum due to the critical need for accurate disease transmission predictions and public health planning. Zhang et al. (2025) developed a comprehensive framework for analyzing an age-structured stochastic SIR model incorporating high-order white noise and Levy jump perturbations, establishing theoretical results on solution existence and uniqueness while deriving stochastic thresholds for disease persistence and extinction using physics-informed neural networks. Complementing this work, Qura Tul Ain et al. (2025) enhanced the classical deterministic SIR model by incorporating soliton-like dynamics and gradient-induced diffusion to capture spatiotemporal disease spread patterns in nomadic populations, revealing the interplay between transmission rates, recovery rates, and advection-diffusion coefficients. Chen & Feng (2025) proposed an age-structured multi-strain epidemiological model using a linearly implicit Euler method that unconditionally preserves biological meaning, demonstrating that numerical dynamics are determined by a basic reproduction number reflecting asymptotic stability. These studies collectively highlight the critical need for robust numerical schemes that can deliver accurate, stable, and efficient solutions across the broad spectrum of problems arising in science, engineering, and public health, a need that the Equally Spaced Numerical Scheme (ESNS) addresses through its power series approximation on uniformly distributed grid points, demonstrating exceptional accuracy and stability in solving both oscillatory differential equations and epidemiological models.

1.1. Study Objectives

The aim of this paper is to study derivation and implementation of an Equally Spaced Numerical Scheme (ESNS) for solving nonlinear problems modeled differential equations. The objectives of the study are to: 1) Derive An Equally Spaced Numerical Scheme (ESNS); 2) Obtain the continuous form of an ESNS; 3) Analyze the numerical properties of an ESNS, that is order and error constant, consistency, convergent, zero- stability and absolute stability region; 4) Carryout the numerical Simulation of an ESNS; and 5) Tabulate and textually shown the comparison of ESNS with existing work.

2.0. Derivation of an Equally Spaced Numerical Scheme (ESNS)

In this section, an equally spaced numerical scheme (ESNS) for solving oscillatory differential equations of the form (1.1). The power series as an approximate solution of the form;

$$\gamma(\xi) = h \sum_{j=0}^{\omega+\varpi-1} \alpha_j \xi^j \quad (2)$$

is consider deriving the method, where ω and ϖ are distinct point of interpolation and collocation.

2.2. Formulation of an Equally Spaced Numerical Scheme (ESNS)

The power series polynomial (2) is considered as an approximate solution of (1). Differentiate (2) once to yield,

$$\gamma'(\xi) = h \sum_{j=0}^{\omega+\varpi-1} j \alpha_j \xi^{j-1} \quad (3)$$

Where $\alpha \in \Re$ for $j = 0\left(\frac{1}{8}\right)_1$ and $\gamma(\xi)$ is continuously differential. Let the solution of (1) be sought on the integration interval $[u, v]$ with a constant step-size h . Substituting equation (2) into (1) gives,

$$f(\xi, \gamma) = h \sum_{j=0}^{\omega+\varpi-1} j \alpha_j \xi^{j-1} \quad (4)$$

Simplify the interpolating point of equation (2) at $\omega = \frac{1}{8}$ and evaluate the collocation point of equation (4) at

$\varpi = 0\left(\frac{1}{8}\right)$ to gives

$$\begin{bmatrix} 1 & \frac{1}{8} & \frac{1}{64} & \frac{1}{512} & \frac{1}{4096} & \frac{1}{32768} & \frac{1}{262144} & \frac{1}{2097152} & \frac{1}{16777216} & \frac{1}{134217728} \\ 0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & \frac{1}{1} & \frac{3}{1} & \frac{1}{1} & \frac{5}{1} & \frac{3}{1} & \frac{7}{1} & \frac{1}{1} & \frac{9}{1} \\ 0 & \frac{1}{h} & \frac{4h}{1} & \frac{64h}{3} & \frac{128h}{1} & \frac{4096h}{5} & \frac{16384h}{3} & \frac{262144h}{7} & \frac{262144h}{1} & \frac{16777216h}{9} \\ 0 & \frac{1}{h} & \frac{2h}{1} & \frac{16h}{3} & \frac{16h}{27} & \frac{256h}{405} & \frac{512h}{729} & \frac{4096h}{5103} & \frac{2048h}{2187} & \frac{65536h}{59049} \\ 0 & \frac{1}{h} & \frac{4h}{1} & \frac{64h}{3} & \frac{128h}{1} & \frac{4096h}{5} & \frac{16384h}{3} & \frac{262144h}{7} & \frac{262144h}{1} & \frac{16777216h}{9} \\ 0 & \frac{1}{h} & \frac{h}{1} & \frac{4h}{3} & \frac{2h}{27} & \frac{16h}{405} & \frac{16h}{729} & \frac{64h}{5103} & \frac{16h}{2187} & \frac{256h}{59049} \\ 0 & \frac{1}{h} & \frac{1}{5} & \frac{75}{1} & \frac{125}{1} & \frac{3125}{1} & \frac{9375}{1} & \frac{109375}{1} & \frac{78125}{1} & \frac{3515625}{1} \\ 0 & \frac{1}{h} & \frac{4h}{1} & \frac{64h}{3} & \frac{128h}{27} & \frac{4096h}{405} & \frac{16384h}{729} & \frac{262144h}{5103} & \frac{262144h}{2187} & \frac{16777216h}{59049} \\ 0 & \frac{1}{h} & \frac{2h}{7} & \frac{147}{1} & \frac{343}{1} & \frac{12005}{1} & \frac{50421}{1} & \frac{823543}{1} & \frac{823543}{1} & \frac{51883209}{1} \\ 0 & \frac{1}{h} & \frac{4h}{1} & \frac{64h}{3} & \frac{128h}{27} & \frac{4096h}{405} & \frac{16384h}{729} & \frac{262144h}{5103} & \frac{262144h}{2187} & \frac{16777216h}{59049} \\ 0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} & \frac{6}{h} & \frac{7}{h} & \frac{8}{h} & \frac{9}{h} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \end{bmatrix} = \begin{bmatrix} \gamma_{n+\frac{1}{8}} \\ f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{bmatrix} \quad (5)$$

Solving for equations (5) for $\{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9, U_{10}\}$ and replacing back into equation (2) to gives a continuous form of schemes as linear multistep method of the form

$$\gamma(\xi) = U_1(\xi) f_{n+\omega} + h \sum_{\varpi}^1 U_i(\xi) f_{n+\varpi}, i = 2, \dots, 10 \quad (6)$$

Where the unknown coefficient $U_1(\xi), U_2(\xi), U_3(\xi), U_4(\xi), U_5(\xi), U_6(\xi), U_7(\xi), U_8(\xi), U_9(\xi), U_{10}(\xi)$ are

$$\left. \begin{aligned} U_1(\xi) &= 1 \\ U_2(\xi) &= -\frac{1070017}{29030400} \tau + \frac{761}{70} \tau^2 + \frac{59062}{945} \tau^3 - \frac{1068}{5} \tau^4 + \frac{34208}{75} \tau^5 - \frac{3072}{5} \tau^6 + \frac{53248}{105} \tau^7 - \frac{8192}{35} \tau^8 + \frac{131072}{2835} \tau^9 \\ U_2(\xi) &= -\frac{2233547}{14515200} + 32\tau^2 - \frac{30784}{105} \tau^3 + \frac{11168}{9} \tau^4 - \frac{673792}{225} \tau^5 + \frac{117760}{27} \tau^6 - \frac{1196032}{315} \tau^7 + \frac{16384}{9} \tau^8 - \frac{1048576}{2835} \tau^9 \\ U_2(\xi) &= \frac{2302297}{14515200} - 56\tau^2 + \frac{3312}{5} \tau^3 - \frac{146824}{45} \tau^4 + \frac{1956992}{225} \tau^5 - \frac{366592}{27} \tau^6 + \frac{3915776}{315} \tau^7 - \frac{278528}{45} \tau^8 - \frac{524288}{405} \tau^9 \\ U_2(\xi) &= -\frac{2797679}{14515200} + \frac{224}{3} \tau^2 - \frac{128192}{135} \tau^3 + \frac{25504}{5} \tau^4 - \frac{1097728}{75} \tau^5 + \frac{72704}{3} \tau^6 - \frac{2441216}{105} \tau^7 + \frac{180224}{15} \tau^8 - \frac{1048576}{405} \tau^9 \\ U_2(\xi) &= \frac{31457}{18144} - 70\tau^2 + \frac{2764}{3} \tau^3 - \frac{46624}{9} \tau^4 + \frac{703552}{45} \tau^5 - \frac{733184}{27} \tau^6 + \frac{1712128}{63} \tau^7 - \frac{131072}{9} \tau^8 - \frac{262144}{81} \tau^9 \\ U_2(\xi) &= -\frac{1573169}{14515200} + \frac{224}{5} \tau^2 - \frac{3008}{5} \tau^3 + \frac{156512}{45} \tau^4 - \frac{2443264}{225} \tau^5 + \frac{2642944}{135} \tau^6 - \frac{6406144}{315} \tau^7 + \frac{507904}{45} \tau^8 - \frac{1048576}{405} \tau^9 \\ U_2(\xi) &= \frac{645607}{14515200} - \frac{56}{3} \tau^2 + \frac{34288}{135} \tau^3 - 1496\tau^4 + \frac{358784}{75} \tau^5 - \frac{26624}{3} \tau^6 + \frac{999424}{105} \tau^7 - \frac{16384}{3} \tau^8 - \frac{524288}{405} \tau^9 \\ U_2(\xi) &= -\frac{156437}{14515200} + \frac{32}{7} \tau^2 - \frac{6592}{105} \tau^3 + \frac{16864}{45} \tau^4 - \frac{274432}{225} \tau^5 + \frac{62464}{27} \tau^6 - \frac{114688}{45} \tau^7 + \frac{475136}{315} \tau^8 - \frac{1048576}{2835} \tau^9 \\ U_2(\xi) &= -\frac{33953}{29030400} - \frac{1}{2} \tau^2 + \frac{242}{35} \tau^3 - \frac{1876}{45} \tau^4 + \frac{30944}{225} \tau^5 - \frac{7168}{27} \tau^6 + \frac{94208}{315} \tau^7 - \frac{8192}{45} \tau^8 - \frac{131072}{2835} \tau^9 \end{aligned} \right\} \quad (7)$$

Equation (7) are expressed as continuous functions of τ as

$$\tau = \frac{\xi - \xi_n}{h} \quad (8)$$

Simplify equation (6) at non interpolating points to gives

$$\gamma_n = \gamma_{n+\frac{1}{8}} - \frac{1070017}{29030400} hf_n - \frac{2233547}{14515200} hf_{n+\frac{1}{8}} + \frac{2302297}{14515200} hf_{n+\frac{1}{4}} - \frac{2797679}{14515200} hf_{n+\frac{3}{8}} + \frac{31457}{181440} hf_{n+\frac{1}{2}} - \frac{1573169}{14515200} hf_{n+\frac{5}{8}} + \frac{645607}{14515200} hf_{n+\frac{3}{4}} - \frac{156437}{14515200} hf_{n+\frac{7}{8}} + \frac{33953}{29030400} hf_{n+1} \quad (9)$$

$$\gamma_n = \gamma_{n+\frac{1}{8}} + h \begin{bmatrix} \frac{33953}{29030400} & \frac{687797}{14515200} & \frac{1622393}{14515200} & -\frac{876271}{14515200} & \frac{8233}{181440} & -\frac{377521}{14515200} & \frac{147143}{14515200} & -\frac{34453}{14515200} & \frac{7297}{29030400} \\ \frac{119}{129600} & \frac{453600}{453600} & \frac{226800}{226800} & \frac{453600}{453600} & \frac{90720}{90720} & \frac{453600}{453600} & \frac{226800}{226800} & \frac{453600}{453600} & \frac{907200}{907200} \\ \frac{369}{358400} & \frac{8101}{179200} & \frac{28809}{179200} & \frac{17217}{179200} & \frac{209}{2240} & \frac{4833}{179200} & \frac{1719}{179200} & \frac{389}{179200} & \frac{81}{358400} \\ \frac{107}{113400} & \frac{359}{8100} & \frac{4681}{28350} & \frac{1154}{14175} & \frac{1927}{11340} & \frac{2201}{56700} & \frac{61}{28350} & \frac{13}{14175} & \frac{13}{113400} \\ \frac{175}{165888} & \frac{26365}{580608} & \frac{93025}{580608} & \frac{55225}{580608} & \frac{5125}{36288} & \frac{75175}{580608} & \frac{34015}{580608} & \frac{2525}{580608} & \frac{425}{1161216} \\ \frac{9}{11200} & \frac{241}{5600} & \frac{477}{2800} & \frac{387}{5600} & \frac{209}{1120} & \frac{387}{5600} & \frac{477}{2800} & \frac{241}{5600} & \frac{9}{11200} \\ \frac{8183}{4147200} & \frac{111587}{2073600} & \frac{261023}{2073600} & \frac{368039}{2073600} & \frac{343}{25920} & \frac{542969}{2073600} & \frac{24353}{2073600} & \frac{408317}{2073600} & \frac{149527}{4147200} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{bmatrix} \quad (10)$$

From equation (9), make $\gamma_{n+\frac{1}{8}}$ the subject formula and substitute the results back to equation (10) to obtained an

Equally Spaced Numerical Scheme (ESNS) of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{n+\frac{1}{8}} \\ \gamma_{n+\frac{1}{4}} \\ \gamma_{n+\frac{3}{8}} \\ \gamma_{n+\frac{1}{2}} \\ \gamma_{n+\frac{5}{8}} \\ \gamma_{n+\frac{3}{4}} \\ \gamma_{n+\frac{7}{8}} \\ \gamma_{n+1} \end{bmatrix} = \gamma_n + h \begin{bmatrix} \frac{1070017}{29030400} & \frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\ \frac{32377}{907200} & \frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{199}{129600} \\ \frac{12881}{358400} & \frac{35451}{179200} & -\frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{368}{358400} \\ \frac{4063}{113400} & \frac{2822}{14175} & -\frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400} \\ \frac{41705}{1161216} & \frac{115075}{580608} & -\frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & \frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888} \\ \frac{401}{11200} & \frac{279}{1400} & -\frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & \frac{333}{1400} & -\frac{29}{5600} & \frac{9}{1400} & -\frac{9}{11200} \\ \frac{149527}{4147200} & \frac{408317}{2073600} & -\frac{24353}{2073600} & \frac{542969}{2073600} & -\frac{343}{25920} & \frac{368039}{2073600} & -\frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200} \\ \frac{989}{28350} & \frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & \frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & -\frac{989}{28350} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{bmatrix} \quad (11)$$

3.0 Analysis of an Equally Spaced Numerical Scheme (ESNS)

The analysis of the basic properties of an Equally Spaced Numerical Scheme (ESNS) shall be analyzed in this section. These analyses are order and error constant, consistency, convergence, zero-stability and region of absolute stability.

3.1. Order and Error Constant of an ESNS

Consider the linear operator $L\{\gamma(\xi); h\}$ associated with the ESNS be defined as,

$$L\{\gamma(\xi); h\} = \alpha_{\frac{1}{8}}(x)y_{n+\frac{1}{8}} + h \sum_{j=0}^1 \beta_j f_{n+j}, \quad j = 0\left(\frac{1}{8}\right)1 \quad (12)$$

Expanding (12) by Taylor series and comparing the coefficients of h yield (Sunday et al. 2015),

$$L\{\gamma(\xi); h\} = c_0 \gamma(\xi) + c_1 h \gamma'(\xi) + c_2 h^2 \gamma''(\xi) + \dots + c_p h^p \gamma^{(p)}(\xi) + \dots c_{p+1} h^{p+1} \gamma^{(p+1)}(\xi) + \dots \quad (13)$$

Definition 1: Let the linear operator defined in equation (13) with the related continuous method (12) are supposed to be of order p if $c_0 = c_1 = c_2 = \dots = c_p = 0$ and $c_{p+1} \neq 0$. c_{p+1} is refers to the error constant and the truncation error is known by,

$$t_{n+k} = c_{p+1} h^{(p+1)}(x_n) + O(h^{p+2}) \quad (14)$$

An equally Spaced Numerical Scheme (ESNS) is expanded on (14) to obtained

$$C_{10} = \begin{bmatrix} 7.3505(-12) & 5.9871(-12) & 6.4964(-12) & 6.1760(-12) \\ 6.4964(-12) & 5.9871(-12) & 7.3505(-12) & 6.1760(-12) \end{bmatrix}^T$$

3.2. Consistency of Method of ESNS

An ESNS is consistent, since the order is more than or equal to one ($p = 9$) (Sunday et al. 2015).

3.3. Zero Stability of ESNS

An ESNS is said to be zero-stable if as $h \rightarrow 0$, the root $z_j, j = 1(1)k$ of the first characteristic polynomial

$\rho(z) = 0$ that is $\rho(z) = \det \left[\sum_{j=0}^k A^{(j)} z^{k-j} \right] = 0$ Satisfies $|z_j| \leq 1$ and for those roots with $|z_j| = 1$, multiplicity must

not exceed two. An ESNS is expressed in the form

$$\rho(z) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = z^7(z-1)$$

Thus, solving for z in

$$\rho(z) = z^7(z-1) = 0 \quad (15)$$

Gives $z_1 = 0, z_2 = 0, z_3 = 0, z_4 = 0, z_5 = 0, z_6 = 0, z_7 = 0, z_8 = 0$ and $z_9 = 1$

Therefore, An ESNS is zero-stable (Delkhosh et al. 2021).

3.4. Convergence of ESNS

Theorem 3.1

The essential condition for ESNS to be convergent is that, it must be consistent and zero stable (Sunday et al. 2015).

Therefore, ESNS is convergent by theorem 3.1.

3.5. Absolute Stability Region of ESNS

To obtain the stability polynomial of an ESNS, we use

$$\bar{v}(w) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} w - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} h - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1070017}{29030400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32377}{907200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12881}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{358400}{4063} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{113400}{41705} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1161216}{401} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11200}{149527} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4147200}{989} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{28350}{28350} \end{bmatrix}$$

$$h \begin{bmatrix} \frac{1070017}{29030400} & \frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\ \frac{32377}{907200} & \frac{113400}{113400} & -\frac{453600}{1719} & \frac{113400}{39967} & -\frac{22680}{351} & \frac{113400}{17217} & -\frac{453600}{7031} & \frac{113400}{243} & -\frac{129600}{368} \\ \frac{12881}{358400} & \frac{179200}{2822} & -\frac{179200}{61} & \frac{179200}{4094} & -\frac{2240}{227} & \frac{179200}{1154} & -\frac{179200}{989} & \frac{25600}{122} & -\frac{358400}{107} \\ \frac{4063}{113400} & \frac{14175}{115075} & -\frac{28350}{3775} & \frac{14175}{159175} & -\frac{2835}{125} & \frac{14175}{85465} & -\frac{28350}{24575} & \frac{14175}{5725} & -\frac{113400}{175} \\ \frac{41705}{1161216} & \frac{580608}{279} & -\frac{580608}{9} & \frac{580608}{403} & -\frac{36288}{9} & \frac{580608}{333} & -\frac{580608}{29} & \frac{580608}{9} & -\frac{165888}{9} \\ \frac{401}{11200} & \frac{1400}{408317} & -\frac{5600}{24353} & \frac{1400}{542969} & -\frac{280}{343} & \frac{1400}{368039} & -\frac{5600}{261023} & \frac{1400}{111587} & -\frac{11200}{8183} \\ \frac{149527}{4147200} & \frac{2073600}{2944} & -\frac{2073600}{464} & \frac{2073600}{5248} & -\frac{25920}{454} & \frac{2073600}{5248} & -\frac{2073600}{464} & \frac{2073600}{2944} & -\frac{4147200}{989} \\ \frac{989}{28350} & \frac{14175}{14175} & -\frac{14175}{14175} & \frac{14175}{14175} & -\frac{2835}{2835} & \frac{14175}{14175} & -\frac{14175}{14175} & \frac{14175}{14175} & -\frac{28350}{28350} \end{bmatrix} w \quad (16)$$

Now, simplify equation (16) and collect the like terms to obtain the stability polynomial as

$$\bar{h}(w) = \left(\frac{417744659}{76710263390208000} w^7 + \frac{1}{150994944} w^8 \right) h^8 + \left(-\frac{761}{2642411520} w^8 + \frac{49763165419}{2013644413992960000} w^7 \right) h^7 + \left(-\frac{6724482907}{12585277587456000} w^7 + \frac{29531}{3963617280} w^8 \right) h^6 + \left(-\frac{89}{655360} w^8 + \frac{5204419680941}{107873807892480000} w^7 \right) h^5 + \left(-\frac{862373358001}{1123685498880000} w^7 + \frac{1069}{589824} w^8 \right) h^4 + \left(-\frac{9}{512} w^8 + \frac{6040298487439}{561842749440000} w^7 \right) h^3 + \left(-\frac{5931801838133}{70230343680000} w^7 + \frac{91}{768} w^8 \right) h^2 + \left(-\frac{1}{2} w^8 + \frac{3179713}{7257600} w^7 \right) h - w^7 + w^8 \quad (17)$$

The stability polynomial (17) is used to plot the absolute stability region of the new method as

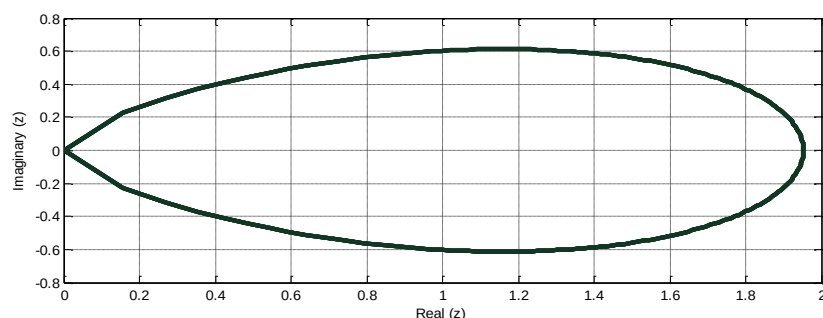


Figure 1. The Stability Region of an ESNS.

The region of absolute stability region in figure 1 is A-stable, because the stability region is the exterior of the contour including the entire negative plane.

4.0. Numerical Simulation of an Equally Spaced Numerical Scheme (ESNS)

An Equally Spaced Numerical Scheme (ESNS) was simulated on oscillatory problematic differential equation of the form (1). On solving the oscillatory class of differential equations, the results are compared with the existing work in literature, specifically, James et al. (2013) and Sunday et al. (2015), Sunday et al. 2013, Omar and Adeyeye, (2016). The numerical comparison is tabulated and graphical shown. The following notations were used in the tables and figures.

notation	Meaning
ξ	Means Evaluation point
EESNS	Means Error in an Equally Spaced Numerical Scheme
ESe15	Means Error in Sunday et al. (2015)
EJe13	Means Error in James et al. (2013)
EOA16	Means Error in Omar and Adeyeye, (2016)
ESe13	Means Error in Sunday et al. (2013)

Problem 1

Consider the oscillatory class of differential equation of the form

$$\gamma(\xi) = \xi - \lambda \quad (18)$$

Whose analytical solution is given by

$$\gamma(\xi) = \xi + \exp(-\xi) - 1 \quad (19)$$

Source: [James et al. (2013) and Sunday et al. (2015)]

Table 1. Numerical Comparison of Problem 1.

ξ	Analytical Solution	Numerical Solution	EESNS	ESe15	EJe13
0.1	0.00483741803595957320	0.00483741803595957316	4.0000e-20	1.0899e-14	1.6554e-11
0.2	0.01873075307798185870	0.01873075307798185867	3.0000e-20	3.6577e-14	4.3981e-11
0.3	0.04081822068171786610	0.04081822068171786606	4.0000e-20	4.4761e-14	7.8451e-11
0.4	0.07032004603563930070	0.07032004603563930073	3.0000e-20	6.1209e-14	1.2662e-11
0.5	0.10653065971263342360	0.10653065971263342357	3.0000e-20	6.1209e-14	1.9709e-10
0.6	0.14881163609402643260	0.14881163609402643258	2.0000e-20	7.0592e-14	3.0180e-10
0.7	0.19658530379140951470	0.19658530379140951464	6.0000e-20	7.9268e-14	4.5771e-10
0.8	0.24932896411722159140	0.24932896411722159136	4.0000e-20	8.3601e-15	6.8954e-09
0.9	0.30656965974059911190	0.30656965974059911181	9.0000e-20	9.4146e-15	1.0336e-09
1.0	0.36787944117144232160	0.36787944117144232169	9.0000e-20	9.7071e-15	1.5435e-09

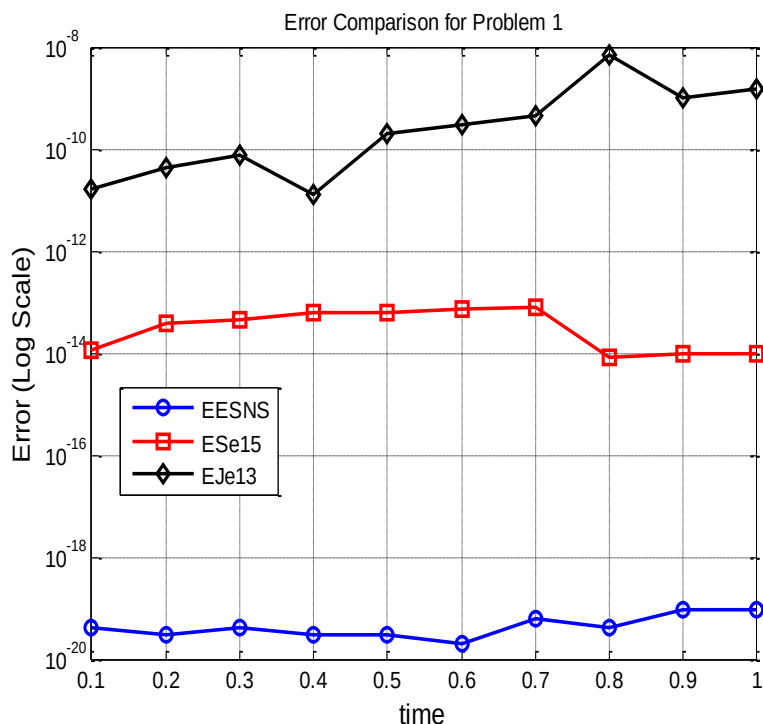


Figure 2. Graphical Curve of Table 1.

Table 1 and Figure 2 collectively present the numerical comparison for Problem 1, an oscillatory differential equation, by displaying the analytical solution, the ESNS numerical solution and error magnitudes of an ESNS alongside those of James et al. (2013) and Sunday et al. (2015) across evaluation points from 0.1 to 1.0. The tabular results demonstrate that the ESNS achieves exceptionally high accuracy, with error values consistently on the order of 10^{-20} , which are substantially smaller than the errors recorded for the existing methods ranging from 10^{-14} to 10^{-9} . This significant reduction in error magnitude underscores the superior precision and computational reliability of the ESNS.

The graphical illustration in Figure 2 reinforces these findings by plotting the analytical and numerical solution curves over the interval from 0 to 1, showing that the ESNS solution completely overlaps the analytical trajectory with no visible deviation, thereby validating the scheme's effectiveness and stability in handling oscillatory differential equations.

Problem 2

Consider the oscillatory class of differential equation of the form

$$\gamma(\xi) = \xi \bullet \gamma, \gamma(0) = 1 \quad (20)$$

Whose analytical solution

$$\xi(\xi) = \exp\left(\frac{1}{2}\xi^2\right) \quad (21)$$

Source: [James et al. (2013) and Sunday et al. (2015)]

Table 2. Numerical Comparison of Problem 2.

ξ	Analytical Solution	Numerical Solution	EESNS	EJe13	ESe15
0.1	1.00501252085940106340	1.00501252085940106340	0.0000e00	1.6554e-11	1.2473e-13
0.2	1.02020134002675581020	1.02020134002675581020	0.0000e00	4.3981e-11	2.4989e-13
0.3	1.04602785990871694270	1.04602785990871694270	0.0000e00	7.8451e-11	4.0149e-13
0.4	1.08328706767495855440	1.08328706767495855440	0.0000e00	1.2662e-11	5.7196e-13
0.5	1.13314845306682631680	1.13314845306682631660	2.0000e-19	1.9709e-10	7.5116e-13
0.6	1.19721736312181016490	1.19721736312181016470	2.0000e-19	3.0180e-10	9.2698e-13
0.7	1.27762131320488661070	1.27762131320488661040	3.0000e-19	4.5771e-10	3.0572e-12
0.8	1.37712776433595708450	1.37712776433595708410	4.0000e-19	6.8954e-09	3.1135e-12
0.9	1.49930250005676686970	1.49930250005676686870	1.0000e-18	1.0336e-09	6.1995e-12
1.0	1.64872127070012814680	1.64872127070012814530	1.5000e-18	1.5435e-09	6.6348e-12

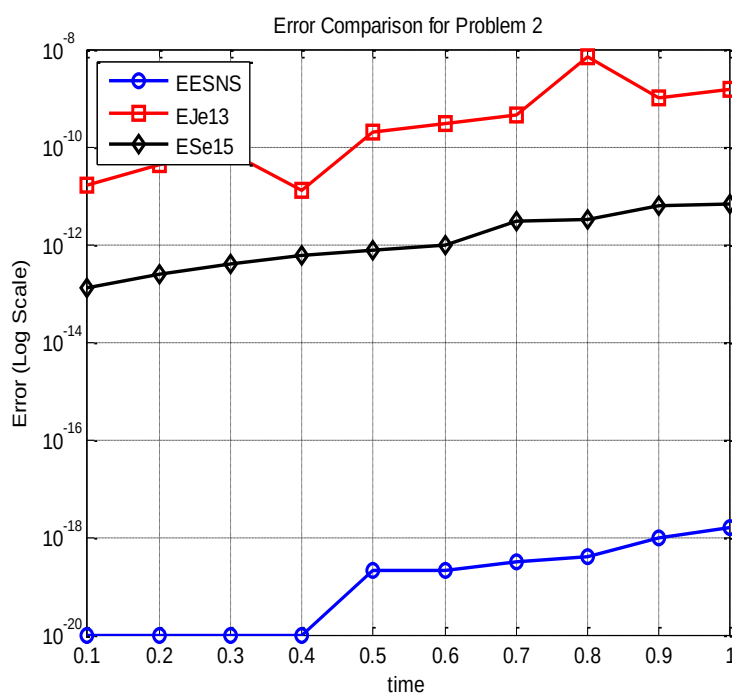

Figure 3. Graphical Curve of Table 2

Table 2 and Figure 3 display the numerical comparison for Problem 2, another oscillatory differential equation, showing the analytical solution, the ESNS numerical solution and error estimates for ESNS compared with those of James et al. (2013) and Sunday et al. (2015) at evaluation points from 0.1 to 1.0. The ESNS achieves remarkable precision, with most error values being zero or as small as 10^{-19} , while the competing methods produce considerably larger errors ranging from 10^{-11} to 10^{-9} , highlighting the ESNS's exceptional ability to approximate the analytical solution with virtually no loss of accuracy. The graphical representation in Figure 3 corroborates these results by illustrating the close agreement between the ESNS approximation and the exact solution over the interval, with the plotted curves aligning seamlessly and showing no visible discrepancies. This combined evidence confirms that the ESNS maintains excellent accuracy and stability throughout the computation, making it a robust and dependable numerical tool for solving oscillatory differential equations.

Problem 3

Consider a susceptible infected recovered (SIR) disease modeled as an epidemical disease which computes the total of number of people that are sick with a communicable infection in a population over a period of time. The differential equation is modelled as

$$\gamma(\xi) = 0.5 - 0.5\xi$$

$$\gamma'(\xi) = 0.5 - 0.5\xi, \gamma(0) = 0.5 \quad (22)$$

Whose analytical solution is

$$\gamma(\xi) = 1 - \frac{1}{2} \exp(-\xi) \quad (23)$$

Source: [Sunday et al. (2013), Omar and Adeyeye, (2016)]

Table 3. Numerical Comparison of Problem 3.

ξ	Analytical Solution	Numerical Solution	EESNS	EOA16	ESe13
0.1	0.52438528774964299546	0.52438528774964299543	3.0000e-20	4.9562e-06	5.5744e-12
0.2	0.54758129098202021342	0.54758129098202021335	7.0000e-20	4.7260e-06	3.9462e-12
0.3	0.56964601178747109638	0.56964601178747109629	9.0000e-20	8.9799e-06	8.1832e-12
0.4	0.59063462346100907066	0.59063462346100907054	1.2000e-19	8.5524e-06	3.4361e-11
0.5	0.61059960846429756588	0.61059960846429756571	1.7000e-19	1.2193e-05	1.9294e-10
0.6	0.62959088965914106696	0.62959088965914106679	1.7000e-19	1.1608e-05	1.8790e-10
0.7	0.64765595514064328282	0.64765595514064328263	1.9000e-19	1.4713e-05	1.7768e-10
0.8	0.66483997698218034963	0.66483997698218034940	2.3000e-19	1.4004e-05	1.7247e-10
0.9	0.68118592418911335343	0.68118592418911335320	2.3000e-19	1.6643e-05	1.8476e-10
1.0	0.69673467014368328820	0.69673467014368328793	2.7000e-19	1.5839e-05	3.0058e-10

Table 3 and Figure 4 present the numerical results for Problem 3, which models a susceptible-infected-recovered (SIR) epidemiological system, comparing the analytical solution, the ESNS numerical solution, and error magnitudes of the ESNS with those of Omar and Adeyeye (2016) and Sunday et al. (2013) across evaluation points from 0.1 to 1.0. The tabular results demonstrate that the ESNS achieves exceptional accuracy with error values consistently around 10^{-19} to 10^{-20} , whereas the existing methods exhibit substantially larger errors ranging from 10^{-6} to 10^{-10} , affirming the scheme's high effectiveness for epidemiological modeling. The graphical comparison in Figure 4 reinforces these findings by showing that the ESNS numerical solution curve closely tracks the analytical solution curve over the entire interval, with the two lines appearing nearly indistinguishable throughout the plotted range. This combined tabular and visual evidence confirms the ESNS's capability to reliably solve complex biological models while maintaining excellent stability and precision, further validating its practical applicability in real-world epidemiological simulations.

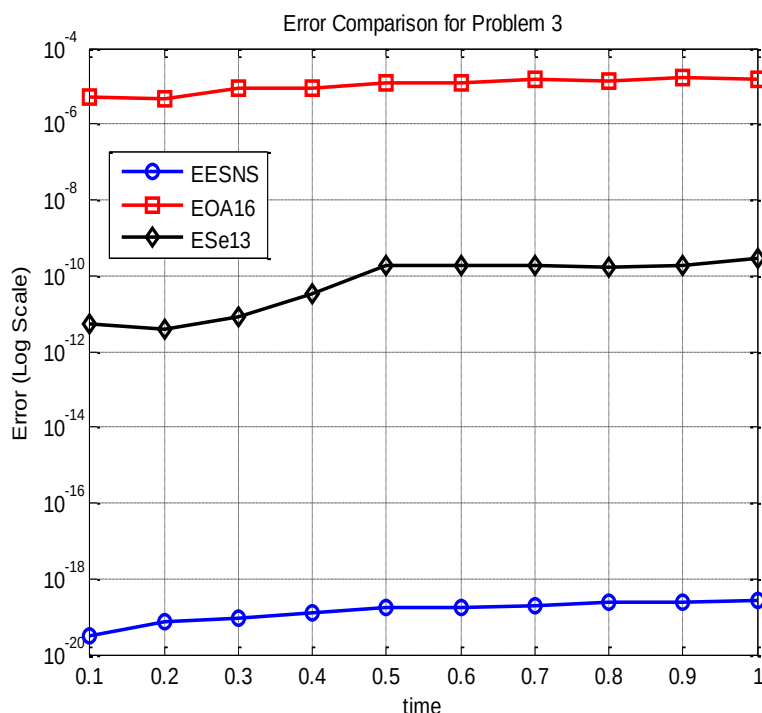


Figure 4. Graphical Curve of Table 3.

5.0. Discussion of Results

The performance of the Equally Spaced Numerical Scheme (ESNS) was examined on oscillatory classes of differential equations represented in Problems 1 to 3. The numerical results obtained were compared with existing methods in the literature, namely James et al. (2013), Sunday et al. (2013, 2015), and Omar and Adeyeye (2016). The comparison was presented using tabular and graphical illustrations, where evaluation points were used to measure the accuracy of the scheme. The error measures were denoted as EESNS for the proposed method, while ESe15, EJe13, EOA16 and ESe13 represent the errors associated with the respective existing methods.

For Problem 1, the numerical results of the ESNS showed excellent agreement with the analytical solution across all evaluation points. The error values of EESNS were extremely small, indicating very high accuracy and computational stability of the scheme. When compared with the methods of James et al. (2013) and Sunday et al. (2015), the proposed scheme consistently produced smaller error magnitudes. The graphical representation in Figure 2 further confirms the closeness between the numerical and analytical solutions, demonstrating the reliability of the scheme in handling oscillatory differential equations.

Similarly, the results obtained for Problem 2 revealed that the Equally Spaced Numerical Scheme maintained remarkable precision in approximating the analytical solution. The numerical and analytical solutions were virtually identical at most evaluation points. In contrast, the competing methods recorded comparatively larger error magnitudes. The graphical curve in Figure 3 shows overlapping trajectories of the numerical and analytical solutions, indicating that the ESNS is highly efficient for solving oscillatory differential equations.

In Problem 3, which involved a susceptible-infected-recovered (SIR) epidemic model, the proposed ESNS method again demonstrated superior accuracy and stability. The error estimates were consistently very small, outdoing the

methods of Omar and Adeyeye (2016) and Sunday et al. (2013). The graphical illustration in Figure 4 confirms that the ESNS solutions closely follow the analytical solution curve. The numerical experiments suggest that the Equally Spaced Numerical Scheme is an effective and reliable approach for solving oscillatory differential equations and epidemiological models due to its high accuracy and computational consistency.

6.0. Summary and Conclusion

This study developed and analyzed an Equally Spaced Numerical Scheme (ESNS) for solving nonlinear oscillatory differential equations where analytical solutions are difficult to obtain. The ESNS was formulated using power series interpolation and collocation techniques with uniform step sizes to improve computational simplicity, stability, and convergence. Theoretical analysis showed that the proposed scheme is consistent, zero-stable, convergent, and possesses an A-stable absolute stability region, making it suitable for stiff and oscillatory initial value problems. Numerical simulations were carried out on three test problems, including oscillatory models and an SIR epidemiological model, and the results were compared with existing methods in the literature.

The numerical experiments demonstrated that the Equally Spaced Numerical Scheme produced highly accurate approximations with extremely small error magnitudes across all evaluation points. When compared with methods by James et al. (2013), Sunday et al. (2013, 2015), and Omar and Adeyeye (2016), the ESNS consistently exhibited superior performance in terms of precision and stability. The graphical and tabular results confirmed that the proposed scheme closely followed the analytical solution curves, indicating strong reliability for oscillatory differential equations. Therefore, the Equally Spaced Numerical Scheme is an efficient computational tool for solving nonlinear oscillatory problems and related mathematical models in science and engineering applications.

7.0. Suggestions for Further Research

Based on this study, the following are suggestions for further research:

1. The current ESNS was developed for single first-order differential equations. Further research should extend the scheme to solve systems of first-order equations and higher-order ODEs directly, broadening its applicability to more complex real-world problems.
2. The ESNS could be adapted as a time-stepping solver for nonlinear PDEs after spatial discretization, opening doors for applications in fluid dynamics, heat transfer, and wave propagation problems.
3. Developing an adaptive version of ESNS with embedded error estimation would automatically adjust step-sizes based on local error, improving computational efficiency for problems with rapidly changing solutions.
4. Although ESNS is proven A-stable, dedicated benchmarking against standard stiff solvers on challenging problems like the Van der Pol oscillator and Kepler problem would further validate its stability and efficiency.
5. Future studies should extend comparative analysis to include recent advanced methods such as Haar wavelet, Bernstein polynomial, block hybrid methods, and spectral methods, evaluating multiple performance metrics to clearly establish ESNS's competitive advantages and limitations.

Declarations**Source of Funding**

This study did not receive grant from any funding agencies in the public, commercial, or not-for-profit sectors.

Competing Interests Statement

The authors declare no competing professional, personal or financial interests.

Consent for Publication

The authors declare that they consented to the publication of this article.

Authors' Contributions

Both the authors made an equal contribution from conceptual design, data collection, drafting the article to revision of the article. Both the authors have read and approved the final copy of the manuscript.

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